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Supergravity M5-branes wrapped on Riemann surfaces and their QFT duals

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Abstract: We find solutions of eleven-dimensional supergravity for M5-branes wrapped on Riemann surfaces. These solutions preserve $\mathcal{N} = 2$ four-dimensional supersymmetry. They are dual to $\mathcal{N} = 2$ gauge theories, including non-conformal field theories. We work out the case of $\mathcal{N} = 2$ Yang-Mills in detail.

Keywords: M-Theory, AdS/CFT Correspondance, Supergravity Models.

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Contents

1. Introduction

The AdS/CFT duality [1, 2] and its generalizations relate supergravity in certain backgrounds to quantum field theories. Most notably $\mathcal{N} = 4$ Yang-Mills has been studied extensively through its dual $AdS_5 \times S^5$ background in type-IIB supergravity. These backgrounds arise as near-horizon geometries of branes on which the quantum field theories live as world-volume theories. Some quantum field theories with lower supersymmetry have been studied by using a number of techniques including orbifolds and orientifolds probed by D3-branes. Others have been constructed through flows of the $\mathcal{N} = 4$ theory by supersymmetry preserving perturbations.[1] In this paper we study duals of $\mathcal{N} = 2$ gauge theories realized as M5-branes wrapped on Riemann surfaces.

In a previous paper, two of us found a solution for a particular Riemann surface which described a set of intersecting M5-branes. In the present work we generalize the construction to all wrapped M5-brane configurations which preserve 8 real supersymmetries (corresponding to $\mathcal{N} = 2$ in four dimensions). One of the advantages of this construction is that given an $\mathcal{N} = 2$ gauge theory described in terms of a Seiberg-Witten Riemann surface one can construct a dual supergravity solution. Thus this is a step towards a systematic classification of geometries dual to $\mathcal{N} = 2$ gauge theories. Another approach to finding solutions for wrapped M5-branes which can be generalized to arbitrary Riemann surfaces was presented in [4].

[1] See [3] for an introduction to these techniques and references.

2. M5-brane setup

We will first describe the M5-brane configurations we want to consider. Much of this is discussed in more detail in [5] and the special case of orthogonally intersecting branes is solved in [6].

We are interested in Hanany-Witten configurations [7] describing $\mathcal{N} = 2$ gauge theories in four dimensions. As shown in [8, 9], the relevant M-theory description is in terms of an M5-brane wrapped on a (non-compact) Riemann surface Σ which is identified with the Seiberg-Witten curve. We want to consider the supergravity description of this system. In particular, the near-horizon limit in supergravity is expected to provide a supergravity dual description of the field theory. This would give a large class of conformal and non-conformal examples of Maldacena's AdS/CFT conjecture [1] and its generalizations.

In our notation, the M5-brane worldvolume is $\mathbb{R}^{3,1} \times \Sigma$ where $\mathbb{R}^{3,1}$ has coordinates x^μ (where μ runs from 0 to 3) and Σ is a holomorphic curve in $\mathbb{C}^2$ with complex coordinates v and s. In terms of real coordinates:

$$v \equiv z^1 = x^4 + ix^5 \,, \tag{2.1}$$

$$s \equiv z^2 = x^6 + ix^7 \,. \tag{2.2}$$

Ten-dimensional type-IIA theory is reached by compactifying x^7 on a circle of radius R. After compactifying, the gauge theory is seen to arise as the worldvolume theory on D4-branes suspended between NS5-branes [9]. The remaining coordinates x^α (where α runs from 8 to 10) are transverse to the M5-brane. The supergravity solution for such a configuration is given by [5]:

$$ds^2 = g^{-1/3} dx_{3+1}^2 + g^{-1/3} g_{m\overline{n}} dz^m dz^{\overline{n}} + g^{2/3} \delta_{\alpha\beta} dx^\alpha dx^\beta \,, \tag{2.3}$$

and the 4-form field strength:

$$\begin{aligned} F_{m\overline{n}\alpha\beta} &= \frac{i}{4} \epsilon_{\alpha\beta\gamma} \partial_\gamma g_{m\overline{n}} \,, \\ F_{m89(10)} &= -\frac{i}{2} \partial_m g \,, \\ F_{\overline{m}89(10)} &= \frac{i}{2} \partial_{\overline{m}} g \,, \end{aligned} \tag{2.4}$$

where $g_{m\overline{n}}$ is required to be a Kähler metric determined by:

$$\partial_\gamma \partial_\gamma g_{m\overline{n}} + 4 \partial_m \partial_{\overline{n}} g = J_{m\overline{n}} \,, \tag{2.5}$$

where J is the source specifying the position of the M5-brane. The square root of the determinant of the Kähler metric is denoted $g = g_{v\overline{v}} g_{s\overline{s}} - g_{v\overline{s}} g_{s\overline{v}}$. For later use we define a Kähler potential $K(w, \overline{w}, y, \overline{y}, t)$ so that $g_{m\overline{n}} = \partial_m \partial_{\overline{n}} K$.

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3. Decoupling limit of wrapped M5-branes

In this section we will describe the near-horizon limit and the equations which must be solved in this limit. This is essentially the same as the limit taken in the special case of orthogonally intersecting M5-branes [6]. We simply take the limit where all field theory quantities (gauge couplings and masses) are fixed while the eleven-dimensional Planck length is taken to zero, $l_P \to 0$. We label the new coordinates, fixed under the scaling, as in [6]:

$$\begin{aligned} w &= \frac{v}{\alpha'} = \frac{vR}{l_p^3}\,, \\ t^2 &= \frac{r}{g_s \alpha'^{3/2}} = \frac{r}{l_p^3}\,, \\ y &= \frac{s}{R}\,, \end{aligned} \tag{3.1}$$

where t is real while w and y are complex. The metric now becomes:

$$\frac{1}{l_p^2} ds^2 = g^{-1/3} \eta_{\mu\nu} dx^\mu dx^\nu + g^{-1/3} g_{m\overline{n}} dz^m dz^{\overline{n}} + g^{2/3}(4t^2 dt^2 + t^4 d\Omega_2^2)\,, \tag{3.2}$$

where now m, n run over y, w and $d\Omega_2^2$ is the metric on the round unit 2-sphere. The source equations become:

$$\frac{1}{4t^5}\partial_t(t^3\partial_t) g_{m\overline{n}} + 4\partial_m \partial_{\overline{n}} g = -\pi^2 \frac{\delta(t)}{t^5} \tilde{J}_{m\overline{n}}(\Sigma)\,, \tag{3.3}$$

where $\tilde{J}_{m\overline{n}}(\Sigma)$ is the source specifying the location of the curve Σ in $\mathbb{C}^2$. The normalisation is such that for a single M5-brane located at $t = y = 0$ we would have $\tilde{J}_{y\overline{y}} = \delta^{(2)}(y)$. The problem is now, given some Σ, to solve the source equations (3.3) for the Kähler metric. The holomorphic curve Σ can be specified as the zero locus of a holomorphic function $f(w, y)$. So the general problem is to find the Kähler potential K in terms of $f, \overline{f}$ and t.

4. Near-horizon geometry of wrapped M5-branes

This paper is based on a set of mathematical identities relevant for solving eq. (3.3). The key observation is the following. Consider a Kähler potential consisting of two terms:

$$K = K^{(1)}(t, F(w, y), \overline{F}) + K^{(2)}(G(w, y), \overline{G})\,, \tag{4.1}$$

where $K^{(1)}(t, F, \overline{F})$ depends on w, y only through the holomorphic function F, and $K^{(2)}(G, \overline{G})$ does not depend on t but depends on w, y only through the holomorphic function G. It is then clear that the Kähler metrics derived solely from $K^{(1)}$ or $K^{(2)}$ have vanishing determinants (since they only depend on the variables through a single holomorphic function each). Thus the determinant of the total metric comes from cross terms.

If we use the following ansatz for $K^{(1)}$:

$$K^{(1)}(t,F,\overline{F}) = \frac{c}{4t^2}\ln\frac{\sqrt{t^4+|F|^4}+t^2}{\sqrt{t^4+|F|^4}-t^2} \tag{4.2}$$

we can solve the differential equation (3.3) as long as the determinant of the metric g is given by the expression:

$$g = \frac{c}{8(t^4+|F|^4)^{3/2}}\,. \tag{4.3}$$

Thus to satisfy the differential equation completely we only need to determine $K^{(2)}$ through the condition for the determinant. This condition results in the following equation:[2]

$$|\partial_y F^2 \partial_w G - \partial_w F^2 \partial_y G|^2 \partial_G \partial_{\overline{G}} K^{(2)} = 1\,. \tag{4.4}$$

To get a relation between F and G which respects their holomorphicity we must have $\partial_G \partial_{\overline{G}} K^{(2)} = |H(G)|^2$, where H is a holomorphic function of G. Hence by appropriately picking G one can cast $K^{(2)}$ in the form:

$$K^{(2)} = |G(w,y)|^2\,, \tag{4.5}$$

with G determined in terms of F by:[3]

$$\partial_y F^2 \partial_w G - \partial_w F^2 \partial_y G = 1\,. \tag{4.6}$$

We turn next to determining F.

We have so far discussed satisfying (3.3) away from any delta function singularities. As it turns out there are delta functions appearing on the right-hand side. These are localized at:

$$t^4 + |F|^4 = 0\,. \tag{4.7}$$

Since both t and $|F|$ are non-negative they must both vanish separately at the delta function singularities. The M5-brane configuration we are interested in is localized at $t=0$ on a Riemann surface. As we are interested in M5-branes wrapped on holomorphic curves it is sensible to pick F to be such that it vanishes on the holomorphic curve.

Let $f(w,y)=0$ be the equation for the holomorphic curve on which we wish to wrap the M5-brane. Let the degree of f in w be denoted by N. Then if we normalize f so that there is no dimensionful parameter multiplying the w^N term, we see that F is determined by dimensional analysis to be

$$F = f^{1/N}\,. \tag{4.8}$$

[2]In fact since the Kähler metric derived from $K^{(2)}$ is independent of t it can be seen that its determinant must vanish to satisfy eq. (4.3) and so it must only depend on a single holomorphic function as we have assumed.

[3]On the right hand side of this equation we can have an arbitrary phase but this can simply be absorbed in G.

4.1 Sources

To completely fix the solution we need to specify the precise form of the sources $\tilde{J}_{m\overline{n}}(\Sigma)$. This will allow us to determine the constant c appearing in the Kähler potential.

As we are considering a single M5-brane wrapping a Riemann surface we have to ensure that $\tilde{J}_{m\overline{n}}(\Sigma)$ has support only on the surface and is normalized such that the total M5-brane charge is 1. These requirements are satisfied by

$$\tilde{J}_{m\overline{n}}(\Sigma) = \delta(f)\partial_m f \partial_{\overline{n}}\overline{f}\,. \tag{4.9}$$

The condition that the M5-brane charge should be 1 is satisfied as long as we integrate once over the "f-plane".

On the other hand when we plug our ansatz for the metric into eq. (3.3) there is a delta function source on the r.h.s. of the equation which needs to be compared to the source (4.9):

$$-\pi^2 \frac{\delta(t)}{t^5}\tilde{J}_{m\overline{n}}(\Sigma) = -c\frac{\pi}{4N}\frac{\delta(t)}{t^5}\delta(f)\partial_m f \partial_{\overline{n}}\overline{f}\,. \tag{4.10}$$

This allows us to fix c using (4.9):

$$c = 4\pi N\,. \tag{4.11}$$

Now the metric is completely determined with the correct normalization.

One can check that this metric agrees with the one presented in [6] when we use the specific (singular) Riemann surface:

$$f = w^N \prod_{i=1}^{n} \sinh(y - y_i)\,, \tag{4.12}$$

which describes localized intersections of M5-branes.

5. The example of SU(N) Yang-Mills

In this section we will obtain the explicit Kähler form for the supergravity solution of M5-brane wrapped on the Seiberg-Witten Riemann surface relevant for SU(N) Yang-Mills [10, 9].

As explained earlier, we have to find a suitable function $G(w, y)$ which satisfies (4.6) for a Riemann surface $f(y, w) = 0$. If we substitute the relation between F and the polynomial representing the surface f and introduce a function h given by, $G = \frac{N}{2} f^{1-2/N} h$ the relation reduces to,

$$df \wedge dh = dw \wedge dy\,. \tag{5.1}$$

In order to express h in a concise manner let us change the independent set of variables from (y,w) to (f,w) . After making this substitution in the above equation (5.1), the equation for $h(f,w)$ becomes

$$\partial_w h(f,w) = -\partial_f y\,, \tag{5.2}$$

while the $\partial_f h(f,w)$ remains undetermined. This is due to the ambiguity in splitting the Kähler form and does not change the metric. For a suitable choice of f this equation can be integrated to obtain a solution for h as

$$h(f,w) = -\int dw \left(\frac{\partial y}{\partial f}\right)(f,w)\,. \tag{5.3}$$

Let us consider an $\mathcal{N}=2$ gauge theory with gauge group $\mathrm{SU}(N)$, corresponding to a pair of parallel NS5-brane with N D4-branes stretched between them [9]. The M-theory lift of this configuration, as mentioned in section 2, will be an M5-brane wrapped on a Riemann surface Σ embedded holomorphically in $\mathbb{C}^2$ with complex coordinates w and y (3.2). The Riemann surface is given by the holomorphic equation:

$$f = e^y + 2B(w) + e^{-y} = 0\,, \tag{5.4}$$

where $B(w)$ is a general polynomial in w of degree N. As is well known the moduli space of this gauge theory is the same as that of the associated Riemann surface.

There is a direct connection between the polynomial $B(w)$ and the gauge theory as described in [10]. The parameters of the gauge theory moduli space s_α can be expressed in terms of the Higgs VEV a_I in the Cartan subalgebra through the relation

$$s_\alpha = (-)^\alpha \sum_{I_1<I_2<\cdots<I_\alpha} a_{I_1}\cdots a_{I_\alpha}\,. \tag{5.5}$$

These parameters occur as the coefficients in the polynomial $B(w)$ as,

$$B(w) = \sum_{\alpha=0}^{n} s_\alpha w^{n-\alpha}\,. \tag{5.6}$$

The function h for the present configuration can be obtained by integrating (5.3) for the surface given by (5.4). Using $\cosh y = \sqrt{(f/2-B(w))^2-1}$ we can write down the solution

$$h(f,w) = -\frac{1}{2}\int_0^w \frac{dw}{\sqrt{(f/2-B(w))^2-1}} \tag{5.7}$$

which can be expressed in a parametric form

$$h(f,w) = -\frac{w}{2}\int_0^1 \frac{dt}{\sqrt{(f/2-B(tw))^2-1}}\,. \tag{5.8}$$

While we have only worked out the $\mathrm{SU}(N)$ Yang-Mills case explicitly, in principle the function G can be found for any Seiberg-Witten curve. All one has to do is solve eq. (4.6).

6. Conclusions and discussion

In this paper we have presented new solutions of eleven-dimensional supergravity. These solutions represent M5-branes wrapped on Riemann surfaces embedded in four-dimensional space. Our solutions preserve 8 real supersymmetries and are dual to $\mathcal{N} = 2$ Seiberg-Witten theories through Maldacena's conjecture.

Our approach allows one, in principle, to find the supergravity dual for *any* Seiberg-Witten theory. In this sense, our approach may be used to classify geometries dual to $\mathcal{N} = 2$ gauge theories.[4]

Although our approach yields the geometry and relevant supergravity quantities we have not tried here to relate them to field theory quantities such as coupling constants, nor have we made any serious attempt at studying the physics of these theories using the supergravity solution. We leave these issues for future work.

One future direction would be to study specific interesting models such as SU(N) Yang-Mills theory, whose supergravity solution is presented in our paper. It would be interesting to compare the large-N behaviour of the supergravity solution to the field theory analysis of Douglas and Shenker [11]. Another question concerns renormalization group flow in these theories, which may be richer and, hopefully, more manageable than the corresponding problem in type-IIB theory with D3-branes. Here one would study the flow of Riemann surfaces into the infrared and determine the corresponding behaviour in the dual field theory.

We hope that our approach will broaden the class of theories amenable to study using Maldacena's conjecture.

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